**Recitation 5**

*Induction*

# Induction

## Why does induction work?

Let's consider an infinite ladder (the best kind of ladder). Suppose we can prove to you both of the following things:

1. You can get to the $1^{\text{st}}$ step of the ladder by stepping up to it.

2. If you can get to the $k^{\text{th}}$ step of the ladder, then you can get to step $k+1$ by stepping up to it.

Why is it the case that for all $n \geq 1$, you can get to the $n^{\text{th}}$ step of the ladder? Discuss with your group.

> We already know we can get to the first step from the first statement. Then, we know we can get to the second step from the second statement. From there, the process repeats and we conclude that we can get to the third, then the fourth... and so on.

Why are we talking about climbing infinite ladders? Well, it turns out this is a good way to think about how induction works.

The *base case* says that we can reach the first step of the ladder.

The *inductive hypothesis* says that we can get to the $k^{\text{th}}$ step of the ladder.

The *inductive step* says that if we can get to the $k^{\text{th}}$ step of the ladder, then we can get to step $k+1$.

Therefore, once we get to step 1, we can get to step 2. Once we get to step 2, we can get to step 3. And so on for all steps of the infinite ladder.

## Induction Template

We will now review the template for an inductive proof.

For example, say we are trying to prove that $\sum_{i=0}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$ is true for all $n \in \mathbb{N}$. In other words, show that this is the equation for calculating the sum of squares $0^2 + 1^2 + 2^2 + \cdots + n^2$.

**Predicate.** Define the predicate $P(n)$. Recall that a predicate is a function that takes in an argument, $n$, and evaluates to true or false.

*Let $P(n)$ be the predicate that*

$$\sum_{i=0}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

**Introduce Induction.** Make the aspirational assertion that, for all $n \geq a$, where $a$ is the smallest value we are considering, $P(n)$ holds. Remember to bound $n$!

*We will show that, for all $n \in \mathbb{N}$, $P(n)$ holds.*

**Base Case.** Show that the base case is true. For some proofs, we may want multiple base cases, but not this time.

*We will first show $P(0)$ is true, that*

$$\sum_{i=0}^{0} i^2 = 0 \quad and \quad \frac{0(0+1)(2*0+1)}{6} = 0$$

*so they are equal.*

**Inductive Hypothesis.** State the inductive hypothesis. In standard induction[1], we assume

$P(k)$ is true for some fixed, arbitrary integer $k \geq a$, where $a$ is your base case value. Sometimes, you may need multiple base cases, and you'll want $k$ to be greater than or equal to the biggest of them.

*Assume $P(k)$ is true for some fixed, arbitrary integer $k \geq 0$.*

**Inductive Step.** Show that $P(k + 1)$ is true given the inductive hypothesis. At some point, you'll want to "invoke the inductive hypothesis", which is using the fact that $P(k)$ is true to show something else in your proof.

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[1]We will also cover strong induction, in which we assume $P(i)$ is true for **all** $a \leq i \leq k$

*We will now show that $P(k+1)$ holds, namely*

$$\sum_{i=0}^{k+1} i^2 = \frac{(k+1)((k+1)+1)(2(k+1)+1)}{6}$$

*We know that*

$$\sum_{i=0}^{k+1} i^2 = \left(\sum_{i=0}^{k} i^2\right) + (k+1)^2$$

*Invoking the inductive hypothesis, we know that*

$$\sum_{i=0}^{k} i^2 = \frac{k(k+1)(2k+1)}{6}$$

*Therefore*

$$
\begin{aligned}
\sum_{i=0}^{k+1} i^2 &= \left(\sum_{i=0}^{k} i^2\right) + (k+1)^2 \\
&= \frac{k(k+1)(2k+1)}{6} + (k+1)^2 \\
&= \frac{k(k+1)(2k+1) + 6(k+1)^2}{6} \\
&= \frac{(k+1)(k(2k+1) + 6(k+1))}{6} \\
&= \frac{(k+1)(2k^2 + k + 6k + 6)}{6} \\
&= \frac{(k+1)(2k^2 + 7k + 6)}{6} \\
&= \frac{(k+1)(k+2)(2k+3)}{6} \\
&= \frac{(k+1)((k+1)+1)(2(k+1)+1)}{6}
\end{aligned}
$$

*as needed.*

**Conclusion.** Conclude your induction.

*Because the base case $P(0)$ holds, and because $P(k) \to P(k+1)$, we have shown by the principle of induction that for all $n \geq 0$, $P(n)$ holds.*    $\square$

## ⋆ **Note** ⋆

For the sake of time, we're only going to look for proof *sketches* in recitation. It's alright to not write down everything, as long as you understand it. In your homework, we'll be looking for full-fledged formal proofs.

## **Task 1**

Jania and Allie are playing a *very fun game.* They have some number of fossils on a table that they can pick from. They take turns picking one, two, or three fossils. Whoever has to pick the last fossil loses. If Allie goes second, prove that she always has a winning strategy if the number of fossils equals $4k + 1$ for some $k \geq 0$.

## **Task 2**

Use induction to prove the following generalization of one of De Morgan's laws:

$$\neg(p_1 \land p_2 \land p_3 \land ... \land p_n) = \neg p_1 \lor \neg p_2 \lor ... \lor \neg p_n$$

for $n \in \mathbb{Z}^+, n \geq 2$.

**Checkpoint 1 — Call over a TA!**

# Strong Induction

With standard induction under our belts, it's time to look at a variant of it, strong induction. In many ways, strong induction is similar to normal induction, as the basic steps listed above are all the same.

Remember that our goal is to prove $\forall i \geq a, P(i)$. The difference is in the inductive hypothesis. When using induction, we assume that $P(k)$ is true to prove $P(k+1)$. In strong induction, we assume that the particular statement holds at all the steps from the base case to the $k$th step. Sometimes, we assume all of $P(b)$, $P(b+1)$, ..., $P(k)$ are true to prove $P(k+1)$, where $b$ is the base case. Note that we may need to prove the base case for multiple values.

Why would we need to do that? Sometimes, you can't just rely on the fact that $P(k)$ is true. Maybe you also need $P(k-1)$ to be true, or perhaps also $P(k-2)$, or even $P(k/2)$. While writing out your inductive step, if you realize that $P(k)$ isn't enough to prove $P(k+1)$, odds are you need strong induction.

# Multiple Base Cases?

Something to think about: If you need both $P(k)$ and $P(k-1)$, you also need *multiple base cases*. Say you're trying to prove $P(n)$ for $n \geq 1$. If you prove $P(1)$ as your base case, how can you show $P(2)$ without $P(0)$ in the inductive step? You'd have to include *both* $P(1)$ and $P(2)$ as base cases.

And so, you naturally might ask...

### Which method should we use?

With some standard types of problems (e.g., sum formulas) it is clear ahead of time what type of induction is likely to be required, but usually this question answers itself during the exploratory/scratch phase of the argument. In the induction step you will need to reach the $k + 1$ case, and you should ask yourself which of the previous cases you need to get there. If all you need to prove the $k + 1$ case is the case $k$ of the statement, then ordinary induction is appropriate. On the other hand, you may realize that you need the two preceding cases ($k - 1$ and $k$) or the full range of preceding cases, to get to $k + 1$, in which case strong induction is needed.

## Example

Prove that every integer $n \geq 2$ can be written as a product of one or more prime numbers.

### Proof

Let $P(n)$ be the predicate "$n$ can be written as a product of one or more prime numbers". We will show that $P(n)$ holds true for any integer $n \geq 2$.

**Base case.** The integer 2 is prime, so it is a product of exactly one prime number (itself). Therefore, $P(2)$ is true.

**Inductive Hypothesis.** Assume the inductive hypothesis, that for a particular integer $k$, $P(i)$ is true for all $2 \leq i \leq k$.

**Inductive Step.** We must prove $P(k+1)$, that $k+1$ is the product of one or more prime numbers. $k+1$ is either prime or composite. If it is prime, then it is the product of exactly one prime number (itself), and $P(k+1)$ is true. If it is composite, then by definition it is the product of two factors, $k+1 = ab$, where $a$ and $b$ are integers $\geq 2$.

Since $a$ and $b$ are both greater than 1, they must also both be less than $k+1$. By the inductive hypothesis, $a$ and $b$ can each by written as a product of one or more primes. But since $k+1 = ab$, we can combine these two products to express $k+1$ as a product of primes, so $P(k+1)$ is true.

Thus **inductive hypothesis and inductive step** imply:

$$\forall k, (\bigwedge_{j=2}^{k} P(j)) \rightarrow P(k+1)$$

**Conclusion.** Since $P(2)$ is true and $P(2), ..., P(k)$ together imply $P(k+1)$, $P(n)$ is true for all integers $n \geq 2$.

## Task 3

1. In the above example proof, why did we only need one base case?

2. In which step of the induction proof is there a difference between ordinary and strong induction? What is the difference?

## Task 4

1. Prove by strong induction that every amount of postage that is at least 15 cents can be made from 4-cent and 5-cent stamps.

   *Hint:* You'll need 4 base cases.

2. Why do we need to use strong induction for this proof? In other words, why can't we just 'assume that $P(k)$ is true' in our induction hypothesis?

3. In part 1, we proved by strong induction that every amount of postage that is at least 15 cents can be made from 4-cent and 5-cent stamps.

   Prove the same claim using **ordinary induction**.

4. *Optional:*  Are strong and ordinary induction equivalent?

## *Optional:* **Task 5**

Consider a candy bar with $n$ squares in a row. Suppose we want to break this candy bar up into individual squares. How many breaks should we perform?



Figure 1: A Delicious Candy Bar

**Claim**: For all $n \geq 1$, **any sequence** of $n - 1$ breaks will reduce a candy bar of $n$ squares into single squares. This means it doesn't matter what order we break squares in: think about this fact for your inductive step!

Prove this claim by (strong) induction.

## Required: Feedback Form

Congrats on finishing Recitation 5—it's remarkable how much we learned about proof techniques, logic, set theory, functions and relations, and induction!

The last part of recitation is to fill out this mandatory feedback form, which should be completed individually. You will be checked off only when the TA confirms that you have submitted (show the "Your response has been recorded" page to the TA).

Your responses will be super helpful as we continue to improve the course, and thank you for an amazing first few weeks of 22!

## Checkoff

If you are done, call over a TA to get checked off. There's a bonus problem below while you wait.

Lastly, just a reminder that you can direct conceptual questions to your TAs during recitation as well!

## *Optional Challenge:* **Green-Eyed Aliens**

There are $n$ (where $n$ is 2 or greater) aliens sitting in a circle so that every alien can see every other alien.

Every alien has green eyes. However, no alien knows its own eye color. Additionally, the aliens cannot talk, so they cannot inform each other of the fact that they have green eyes. However, if an alien ever figures out their own eyes are green, they will leave the spaceship that night.

On Day 1, Professor Lewis comes and tells the circle of aliens that at least one of them has green eyes. Prove that, on the $n$th night, all $n$ aliens will leave the ship.