Problem 1

An interplanetary spaceflight map can be represented as a graph $G = (V, E)$, where $V$ is the set of all planetary spaceports and $(u, v) \in E$ if there is a spaceflight from $u$ to $v$. Suppose Pluto simulates a random spaceflight map of the solar system where, for each pair of spaceports, there is a spaceflight available between them (in either direction) with probability $p$.

a. Let random variable $F$ be the total number of spaceflights. Find the expected value of $F$, justifying your answer.

b. Pluto wants to go on a tour of the solar system. Let random variable $H$ be the number of Hamiltonian tours in $G$. Find the expected value of $H$, justifying your answer.

Note: A tour is defined by the order of the cycle rather than the order from some arbitrary starting vertex. Hence, the tour $(a, b, c, a)$ is the same tour as $(b, c, a, b)$.

c. Define the random variable $T$ to be the number of triplets of spaceports $(a, b, c)$ such that there is a path of exactly 2 spaceflights from $a$ to $c$ through $b$. This means that a spaceflight exists from $a$ to $b$ and from $b$ to $c$, but not from $a$ to $c$ directly. Find the expected value of $T$, justifying your answer.

Hint: Use indicator random variables!
Problem 2

Using strong induction, prove that if $G$ is a simple graph with $n$ vertices, $k$ connected components, and no cycles, then $G$ has $n - k$ edges.

Note: Remember to use build-down induction! This induction proof requires a proof on two variables. Define the predicate $P(n, k)$ where $n \geq k$ and in the inductive step show that both $P(i + 1, j)$ and $P(i, j + 1)$ are true (an increment on both variable independently!).

Hint: If the connected components have no cycles, what must they be?

Problem 3

Define a uniformly-random $k$-coloring $f : V(G) \rightarrow \{1, 2, \ldots, k\}$ as a coloring of $G$ (where $|V(G)| = n$) where each vertex $v \in V(G)$ is assigned one of the $k$ colors uniformly at random. This might produce either a proper or improper coloring of $G$. A graph is properly colored if each vertex in the graph is assigned a color such that for all edges $(u, v)$, $u$ and $v$ are assigned different colors. We say a graph is $n$-colorable if there exists a way to properly color a graph using $n$ colors.

a. In terms of $n$ and $k$, how many $k$-colorings of $G$ are there? Include both proper and improper colorings and justify your answer.

b. Let $K_n$ be the complete graph on $n$ vertices, with $n \geq 3$.

i For what values of $k$ do there exist proper $k$-colorings of $K_n$?

ii What is the probability that a uniformly-random $k$-coloring $f$ is a proper $k$-coloring of $K_n$? You need only consider $k$’s such that a proper $k$-coloring of $K_n$ is possible.

c. For $n \geq 1$, define the graph Hypercube$_n$ as follows. The vertex set is $\{0, 1\}^n$ and, for binary strings of length $n$ denoted by $u$ and $v$, $\{u, v\}$ is an edge of Hypercube$_n$ if and only if $u$ and $v$ differ in exactly one position. For example, Hypercube$_1$ is a single edge, Hypercube$_2$ is a square, and Hypercube$_3$ is a 3-dimensional cube! Note: If it aids your understanding, try drawing these small examples! The name may become clearer to you at that point.

What is the minimum number of colors $k$ that are needed to properly $k$-color Hypercube$_n$?
Mind Bender (Extra Credit)

Recall the graph coloring definitions from Problem 3. We say a graph is \(n\)-colorable if there exists a way to properly color a graph using \(n\) colors.

A graph is planar if it can be drawn on a plane in such a way that its edges intersect only at vertices. In other words, we can draw the graph on a piece of paper in such a way that no two edges overlap.

The 4-color theorem says that any planar graph can be properly colored using only 4 colors. This theorem is famously difficult to prove... but you can prove the 6-color theorem right now! Let \(G = (V, E)\) be a simple, connected, planar graph on at least three vertices. We will be using this graph for the rest of the problem.

a. Recall from lecture that \(\sum_{v \in V} \deg(v) = 2|E|\). Show that the average (mean) degree of vertices in \(V\) is strictly less than 6.

   **Hint:** You may use without proof that, for any simple, connected, planar graph on at least 3 vertices, \(|E| \leq 3|V| - 6\).

b. Show by contradiction that there exists a vertex \(v \in V\) such that \(\deg(v) \leq 5\).

c. Prove, by induction on the number of vertices, that \(G\) is 6-colorable.

   **Note:** Remember to use build-down induction!