Homework 4
Due: Monday, June 14th at 11:59pm EDT

All homeworks are due the Monday after their release at 11:59pm EDT on GradeScope.

Please do not include any identifying information about yourself in the handin, including your Banner ID.

Be sure to fully explain your reasoning and show all work for full credit. We recommend checking out the sample proofs on the course website to see the level of rigor for which we’re looking.

Problem 1

Note: For the purposes of this homework, we will also use the relation symbol \( R \) to represent the graph of the relation \( R \). Thus, \( R \) can be used to express relation between two elements, such as \( a R b \), but can also be used to express the graph of the relation, such as \( R = \{(1, 0), (0, 1)\} \).

a. Which of the following relations are symmetric and which are antisymmetric? Prove your answers.
   
   i. \( R_1 = \{(a, b) \mid a = b \text{ or } a = -b\} \)
   
   ii. \( R_2 = \{(a, b) \mid a = b + 1\} \)

b. Consider the following relations on \( \mathbb{R} \). State if each relation is a function, a partial function, or not a function, and briefly explain your reasoning. Then:
   
   - If it is a function, state if it is injective, surjective, or neither.
   - If it is a partial function, modify the domain to make it a function.
   - If it is not a partial function, briefly explain why.
   
   i. \( R = \{(2, 3), (3, 4), (4, 5), (5, 6), (6, 7)\} \).
   
   ii. \( f(x) = x^2 + 1 \).
   
   iii. \( f(x) = y \text{ if and only if } x^2 = y^2 \).
   
   iv. \( f^{-1}(x) \text{ where } f(x) = x^5 + 40 \).
   
   v. \( f(x) = \sqrt{x} \).

c. Prove that if a relation \( R \) on a set \( S \) is reflexive and transitive, then the relation \( R \cup R^{-1} \) is an equivalence relation, where \( R^{-1} \) is the inverse.
Problem 2

Let $R$ be an equivalence relation on a finite set $A$, where $|A| = n$ and $n$ is a non-negative integer.

a. What is the minimum cardinality of $R$, in terms of $n$? Justify (prove) why you can achieve this cardinality and why no smaller cardinality can be achieved.

b. What is the maximum cardinality of $R$, in terms of $n$? Justify (prove) why you can achieve this cardinality and why no larger cardinality can be achieved.

c. Suppose $n$ is even. What can you say about the parity of $|R|$? Justify (prove) your response.

d. Suppose $n$ is odd. What can you say about the parity of $|R|$? Justify (prove) your response.

Problem 3

a. i. Find all relations with domain $\{0, 1\}$ and co-domain $\{1\}$.
   ii. Find all relations with domain $\{1\}$ and co-domain $\{0, 1\}$.

b. i. Find all functions from $\{0, 1\}$ to $\{1\}$.
   ii. Find all functions from $\{1\}$ to $\{0, 1\}$.

c. Create a bijection between the set $\mathbb{Z}^+$ and the set $\{ n \mid n = 2k \text{ for } k \in \mathbb{Z}^+ \}$ whose inputs and outputs are the binary representation of integers.

Mind Bender (Extra Credit)

a. Consider the relation $R$ on a finite set $S$ where $S \subseteq \mathbb{Z}$. Define $R$ such that $\forall a, b \in S$, if $a > b$ or $a < b$ then $(a, b) \in R$. Prove whether or not the complementary relation $\overline{R}$ is reflexive, irreflexive, symmetric, anti-symmetric, or transitive. Assume the complement is taken relative to the universal set $U = S \times S$.
   Note: you must include a proof for each property.

b. Suppose now that the relation $\preceq_1$ on set $S$ and relation $\preceq_2$ on set $T$ have the same properties as the relation $\overline{R}$ from part (a). If the relation $\preceq$ on the set $S \times T$ is defined by $(s, t) \preceq (u, v)$ if and only if $s \preceq_1 u$ and $t \preceq_2 v$ for $s, u \in S$, and $t, v \in T$, show that $\preceq$ also has the same properties as $\overline{R}$. 