Homework 5
Due: Monday, June 21st at 11:59pm EDT

All homeworks are due the Monday after their release at 11:59pm EDT on Gradescope.

Please do not include any identifying information about yourself in the handin, including your Banner ID.

Be sure to fully explain your reasoning and show all work for full credit. We recommend checking out the sample proofs on the course website to see the level of rigor for which we’re looking.

Problem 1

a. Prove that, provided two bijections $f : Q \rightarrow R$ and $g : R \rightarrow Q$ (where $Q$ and $R$ are non-empty), the composition $f \circ g$ is a bijection.

b. Applying an arbitrary function repeatedly to some initial value is called an iterated map. Let $f$ be a bijection $f : R \rightarrow R$. Let $g^n$ be a function $g^n : R \rightarrow R$ that applies $f$ a total of $n$ times where $n \geq 1$. That is, $g^n(x) = f(f(f(...f(x)...)))$ where $f$ is applied $n$ times. Prove that $g^n$ is a bijection using induction.

Problem 2

The two moons of Mars, Phobos and Deimos have been stuck in tidally locked orbits for millions of years. The same face of each moon points towards the planet at all times. In an effort to make staring at the surface a little more exciting, they decide to play a very fun game.

Phobos gathers incoming Martian meteors and places them on the Martian surface into two separate piles of equal size. Then, at each turn Phobos and Deimos each remove some positive number of meteors from one of the piles. The player who removes the last meteor wins. Since the game is Phobos’ idea, it decides to go first. Prove that Deimos can always win.

Hint: Use strong induction!
Problem 3

a. Prove by induction that for all positive integers $n$, there exists a positive integer $m$ such that:

$$m^2 \leq n < (m + 1)^2$$

b. Prove by contradiction that there exists a **unique** such $m$.

Mind Bender (Extra Credit)

a. Consider the figure below.

![Square A](image)

How does the number of non-overlapping squares in the figure change if Square A is subdivided into squares whose side lengths are half that of the original’s? To give you a sense of what counts as a non-overlapping square, the figure initially has 15 non-overlapping squares.

b. Divide the square below into exactly 6 (not necessarily congruent) squares such that there is no overlap and the entire region is covered.

![Square](image)

c. Prove by induction that every integer greater than 5 can be written in the form $a + 3m$, where $a$ is 6, 7, or 8, and $m$ is a non-negative integer.
d. Using parts (a), (b), and (c) for inspiration, prove by induction that a square can be subdivided into $n$ (not necessarily congruent) squares such that there is no overlap and the entire region is covered for all integral $n$ greater than 5.

**Note:** You may assume that your answer to (a) is correct.