All homeworks are due the Monday after their release at 11:59pm EDT on Gradescope.

Please do not include any identifying information about yourself in the handin, including your Banner ID.

Be sure to fully explain your reasoning and show all work for full credit. We recommend checking out the sample proofs on the course website to see the level of rigor for which we’re looking.

Problem 1

In this problem, you will prove the main steps of an alternate proof of Fermat’s Little Theorem. We will work with an equivalent version of Fermat’s Little Theorem, which states that for all integers \( a \) and prime numbers \( p \), \( a^p \equiv a \pmod{p} \).

a. Prove that, for all prime numbers \( p \) and all integers \( 0 < i < p \), \( p \) divides \( \binom{p}{i} \).

b. Prove that, for integers \( a \) and \( b \) and a prime number \( p \), \( (a + b)^p \equiv a^p + b^p \pmod{p} \).

c. Complete the proof of Fermat’s Little Theorem for positive integers using induction on \( a \).

Problem 2

Let \( p_1, p_2, \) and \( p_3 \) be distinct primes and let \( r_1, r_2, \) and \( r_3 \) be positive integers.

a. Let \( n = p_1^{r_1} p_2^{r_2} \).

i. How many integers between 1 and \( n \) (inclusive) are divisible by \( p_1 \)?

ii. How many integers between 1 and \( n \) (inclusive) are divisible by \( p_2 \)?

iii. How many integers between 1 and \( n \) (inclusive) are divisible by both \( p_2 \) and \( p_1 \)?

iv. Hence, how many integers between 1 and \( n \) (inclusive) are divisible by either \( p_1 \) or \( p_2 \) (or both)? Explain the reasoning behind your answer.
b. Consider a positive integer $n$ whose prime factorization features three distinct primes such that $n = p_1^{r_1} p_2^{r_2} p_3^{r_3}$. How many integers between 1 and $n$ (inclusive) are relatively prime to $n$? Explain the reasoning behind your answer.

c. Hence, what is the value of $\varphi(450)$? Explain the reasoning behind your answer.
   **Note:** Recall that $\varphi$ is the symbol for Euler’s totient function.

**Problem 3**

Consider again Saturn’s 82 moons. Each of them has some non-negative, integer number of craters. Prove that it is possible to choose 9 of them whose total number of craters is divisible by 9.

**Hint:** Assign each moon to its number of craters’ congruence class and use the Pigeonhole Principle.

**Mind Bender (Extra Credit)**

Define a set of positive integers to be *yummy* if it contains its own cardinality. A set is *minimally yummy* if it is yummy and it has no proper subset which is also yummy. Count the number of minimally yummy subsets of $\{1, 2, \ldots, n\}$.

**Note:** A proper subset of a set $S$ is a subset of $S$ that is missing at least one member of $S$. 