Problem 1

For each of the following pairs of events, identify whether they are independent. Justify why or why not by using the fact that event $A$ and $B$ are independent if $P(A) = P(A|B)$.

a. When flipping a fair coin three times:
   - The first coin comes up tails ($T$)
   - There is a run of exactly two heads (that is, there are two, but not three, heads flipped in a row)

b. When generating a 0/1 string of length 5:
   - The 3rd bit is a 1
   - There are at least two 0s

Problem 2

The 14 moons of Neptune work at Neptune’s Intergalactic Inc. Each of them works at their respective desk. Triton, one of Neptune’s worker moons, wants to prank Neptune, the boss of the company. Knowing Neptune hates disorder, Triton convinces everyone in the office to switch desks when Neptune is in the boss’ office. The prank is only successful if there is exactly one moon per desk after the jumble and the new arrangement is not identical to the previous one.

a. If each moon chooses a desk uniformly and independently (potentially choosing their own desk), provide an expression for the probability the prank is successful
(that is, each desk is assigned to exactly one moon and at least one moon has moved desks).

b. Now, consider a case in which all desks are arranged in a circle, and the moons only have time to move to a desk directly to the left or right of their current desk. What is the probability the prank is successful? (Every moon moves desks this time.)

c. Neptune is taking a little longer than usual to come out of its office—the moons will move to any desk exactly 2 to the right or left of their original position. In this case, what is the probability that the prank is successful?

**Problem 3**

Neptune is given the following equation:

\[ x_1 + x_2 + x_3 + x_4 = 75 \]

where each \( x_i \) must be non-negative.

a. Count the number of solutions to this equation.

**Hint:** Use Donuts and Separators!

b. Now suppose we require a solution with \( x_1 \) and \( x_3 \) strictly positive. Count the number of solutions under this new constraint.

c. Let us now assume that each \( x_i \) for \( i \in [1, 4] \) takes some value in the range \([0, 75]\) uniformly at random. Given that \( x_1 \) and \( x_3 \) are strictly positive, what is the probability that \( x_1 + x_2 + x_3 + x_4 = 75 \)?

**Mind Bender (Extra Credit)**

Neptune is working on a research paper but its only local planetary library has made it so that each member can only take out a single book. In front of Neptune is an aisle of \( n \) different books relevant to its paper, where \( n \) is a positive integer.

Neptune doesn’t have enough time to go through every book, so it wants to select a book uniformly at random. In other words, Neptune looks to find a scheme that will guarantee that it selects the \( i \)th book with probability \( \frac{1}{n} \), where \( 1 \leq i \leq n \).

Neptune has some constraints. The planet is too tired to make multiple trips up and down the aisle, so it looks to go down the aisle only once. Furthermore, Neptune can only hold one book at a time.
Come up with and prove a scheme that allows Neptune to walk away from the end of the aisle with the $i^{th}$ book with probability $\frac{1}{n}$.

**Note:** We do not know the value of $n$, only that $n$ is a positive integer. Hence, the scheme you create may not assume that the value of $n$ is known.