Random Variables and Expectations

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Overview

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Numerical Values of Outcomes

Sometimes it makes sense to attach a numerical value to the outcome of a probability space.

Example: We ask people to name all the planets they can.

- Earth, Mars
- Earth, Neptune, Mars, Saturn
- Mercury, Venus, Earth, Mars, Jupiter, Saturn, Neptune, Uranus
- Mars, Jupiter

How many outcomes? $2^8 = 256$, assuming we ignore any fictional planets they name like “Tatooine” or “Pluto”.

If we want to summarize the results, we might assign each outcome a statistic, that is, a numerical summary. A natural choice is the number of planets they named: 2, 4, 8, 2.
Random Variable

**Definition:** A random variable $R$ on a probability space is a function whose domain is the sample space.

Example: Let’s say the sample space is a deck of cards and $R$ maps a number card to its value and a face card to 10 and ace to 1. So, $R(2\heartsuit) = 2$ and $R(J\spadesuit) = 10$.

Typically, codomain of $R$ is subset of reals. A random variable is used kind of like a variable, but it is “implemented” as a function.
Coin example

We flip 3 fair coins. Let $C$ be the random variable that is the number of coins that come up heads. Let $M$ be a random variable that is 1 if all three coins come up heads or all three coins come up tails and 0 otherwise. They are random variables in that they map all possible outcomes to values, integers in this case.

Example: $C(THH) = 2$. $M(THH) = 0$. $C(TTT) = 0$. $M(TTT) = 1$.

$C$ is counting the number of heads, $M$ tells us whether or not all the coins match.
In terms of sample space

\[ S = \{ \text{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT} \} \]

\[
\begin{align*}
C(\text{HHH}) &= 3 & C(\text{THH}) &= 2 \\
C(\text{HHT}) &= 2 & C(\text{THT}) &= 1 \\
C(\text{HTH}) &= 2 & C(\text{TTT}) &= 1 \\
C(\text{HTT}) &= 1 & C(\text{TTT}) &= 0 \\
M(\text{HHH}) &= 1 & M(\text{THH}) &= 0 \\
M(\text{HHT}) &= 0 & M(\text{THT}) &= 0 \\
M(\text{HTH}) &= 0 & M(\text{TTT}) &= 0 \\
M(\text{HTT}) &= 0 & M(\text{TTT}) &= 1
\end{align*}
\]
Definition

**Definition**: An *indicator random variable* is a random variable that maps every outcome to either 0 or 1. Indicator random variables are also called *Bernoulli variables*.

Example: The random variable $M$. It “indicates” whether the three coins match.

Connection between indicator random variables and events. Recall, an event is a subset of the sample space—a set of outcomes. An indicator random variable can be interpreted as a set, since it maps each outcome to whether it is *in* the set (1) or *out* of the set (0).

If $E$ is an event, we can define the corresponding indicator random variable $I_E$, where $I_E(\omega) = 1$ if $\omega \in E$ and 0 otherwise.

Example: If we take $E$ to be the event where all 3 coins match, $M = I_E$. 
Partitioning outcome space

An indicator random variable partitions outcome space:

\[
\begin{align*}
&\text{HHT HTH HTT THH THT TTH} \\
&M=0
\\
&\text{HHH TTT} \\
&M=1
\end{align*}
\]

So does any other random variable:

\[
\begin{align*}
&\text{TTT} \\
&C=0
\\
&\text{HHT HTH THH} \\
&C=2
\\
&\text{HHH} \\
&C=3
\end{align*}
\]
Statements about random variables

Each block is a subset of the sample space and therefore an event.
The assertion that $C = 2$ defines an event: $\{THH, HTH, HHT\}$.
$\Pr[M = 1] = 1/4$.

Statements about random variables can also be viewed as events.
$\Pr[C \leq 1] = 1/2$.
$\Pr[M \cdot C \text{ is odd}] = 1/8$.

This last statement is a funny way of saying “all heads”. Why?
Concept

The *expected value* (often *expectation*) of a random variable is its mean or probability weighted average.

Example: Define a random variable $R$ to be the alphabetic position of the first letter of the outcome of a coin flip, $R(H) = 8$, $R(T) = 20$. The expected value of $R$ is 14. It is $1/2 \times 8 + 1/2 \times 20$.

We write $\mathbb{E}[R] = 14$. (Book uses “Ex”, but I can’t pretend that’s ever used.)

Suppose we select a student uniformly at random from the class, and let $R$ be the student’s homework 2 score. Then, $\mathbb{E}[R]$ is just the class average. The expected value is a useful thing to know.
**Definition**

*Definition*: If $R$ is a random variable defined on a sample space $S$, then the expectation of $R$ is

\[ \mathbb{E}[R] := \sum_{\omega \in S} R(\omega) \Pr[\omega]. \]

**Example:**

\[ \mathbb{E}[C] = \frac{0+1+1+1+2+2+2+3}{8} = \frac{3}{2}. \]

**Example:**

\[ \mathbb{E}[M] = \frac{1+0+0+0+0+0+0+1}{8} = \frac{1}{4}. \]

**Exercise for the reader:** If $E$ is an event, $\Pr[E] = \mathbb{E}[I_E]$. 
Do a die

Let $R$ be the random variable corresponding to a fair die. Here, the outcomes are numbers, so we’ll just define $R(\omega) = \omega$.

$$E[R] = \frac{1+2+3+4+5+6}{6} = 7/2 \text{ or } 3.5.$$ 

Does that mean we expect the die to come up 3.5? No, it will never come up 3.5. Maybe “expected value” was a bad choice of name. Shrug.

In general, if $R$ is a random variable with a uniform distribution over $[1, n]$, $E[R] = \sum_{i=1}^{n} i \cdot \frac{1}{n} = \frac{1}{n} \sum_{i=1}^{n} i = \frac{1}{n} n(n+1)/2 = (n+1)/2$. 

One and die

Let $R$ again be the random variable corresponding to a fair die.

\[ 1 + E[R] = 1 + 3.5 = 4.5. \]

\[ E[1 + R] = \frac{2+3+4+5+6+7}{6} = \frac{27}{6} = \frac{9}{2} \text{ or } 4.5. \]

Sometimes the expectation of a function matches the function of the expectation.
One over die

Let $R$ again be the random variable corresponding to a fair die.

$$\frac{1}{\mathbb{E}[R]} = \frac{1}{3.5} = \frac{2}{7} \text{ (.29 ish)}. $$

$$\mathbb{E}\left[\frac{1}{R}\right] = \frac{1+1/2+1/3+1/4+1/5+1/6}{6} = \frac{49}{120} \text{ (.41 ish)}. $$

Sometimes the expectation of a function matches the function of the expectation. Sometimes not.