Please read in full:

You are not allowed to use the internet outside of the CSCI0220 website. Doing so is a breach of the Collaboration Policy and Brown’s Academic Code.

You can refer to lecture slides, lecture captures, your own notes, the course textbook, the website resources, and Campuswire. You must complete the entire exam by yourself, without discussion of any kind with anyone else.

By submitting this exam, you are confirming that you have understood these policies, and that you have followed them.

As always, please do not include any identifying information about yourself in the handin, including your Banner ID. Be sure to fully explain your reasoning and show all work for full credit.

Problem 1

a. Prove using the set-element method that for finite sets $X$ and $Y$ with the same universal set $U$

$$X \cap Y = (X \cup Y) - (\overline{X} \cup \overline{Y}).$$

b. i. Consider an arbitrary binary operation $\star$ on a finite set $X$. A left identity element for $\star$ on the set $X$ is any $e \in X$ such that $e \star x = x$ for all $x \in X$. For example, 1 is the left identity element for the multiplication operator ‘$\times$’ on the integers $\mathbb{Z}$ because $1 \times x = x$ for all $x \in \mathbb{Z}$.

Find the left identity elements of the set difference operator ‘$-$’ on the set $\mathcal{P}(S)$, where $S$ is an arbitrary non-empty, finite set. If there exists one, prove why the element you chose is an identity element, and prove why no other element would be. If there does not exist one, prove that no element can be.

ii. By the same logic, a right identity element for $\star$ on the set $X$ is any $e \in X$ such that $x \star e = x$ for all $x \in X$. For example, 1 is the right identity element for the division operator ‘$/$’ on the integers $\mathbb{Z}$ because $x/1 = x$ for all $x \in \mathbb{Z}$.

Find the right identity elements of the set difference operator ‘$-$’ on the same set $\mathcal{P}(S)$. If there exists one, prove why the element you chose is an identity element, and prove why no other element would be. If there does not exist one, prove that no element can be.
Problem 2

Let $B$ be the set of all possible propositional formulas. Note that $B$ is a set of infinite cardinality. Define the relation $R$ on $B$ such that, for $a, b \in B$, $a \mathrel{R} b$ if and only if $a$ and $b$ always return the same output given the same input.

a. Let $x$ be a proposition with Boolean inputs $p, q$ and $r$ and the following truth table:

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Write a propositional formula corresponding to $x$ in conjunctive normal form with a maximum of two clauses.

Note: Explain how you arrived at your answer.

b. Prove that $R$ is an equivalence relation.

c. Create a bijection between the equivalence classes of three-input, one-output propositional formulas and binary strings of length eight.

Note: You need not prove your answer. You must just explain the reasoning behind your bijection.

d. How many equivalence classes of three-input, one-output propositional formulas are there?

Note: You must thoroughly explain your answer, but are not required to prove it.

e. The equivalence class of a propositional formula $x$ is $[x]_R = \{ a \in B \mid (a, x) \in R \}$.

Prove that $|[x]_R|$ is infinite for any $x \in B$.

Problem 3

Let the relation $R_{15} = \{ (a, b) \mid \exists i \in \mathbb{Z} \text{ such that } (a - b) = 15i \}$. Prove by induction that, for any finite set $S$ with more than one element, $(|\mathcal{P}(\mathcal{P}(S))|, 1) \in R_{15}$.